Apples and Oranges? Comparing Quantum and Classical Theories

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Classical vs Quantum

How to understand quantum physics?

> ” What is quantum mechanics like, you ask?
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> Well, you know classical theories, right?
> Then let’s try to explain quantum mechanics by comparing and contrasting it with classical theories!

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Wait! Do we actually **know** classical theories?

Classical "vibes": some examples

(Local / Macro / …) **Realism**

> A system with two or more distinct states available to it will at all times be in one or the other of these states. [1]

Non-invasiveness (of measurement)

> It is possible, in principle, to determine the state of the system with arbitrarily small perturbation on its subsequent dynamics. [1]

Non-contextuality

> Observables may be simultaneously assigned definite values. [2]

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How can we be more concrete? Look for a **meta-theory** of classical theories...

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The Bell's question

> Is quantum mechanics a classical theory in disguise?

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> Consider a pair of spin one-half particles formed somehow in the singlet spin state and moving freely in opposite directions. Measurements can be made, say by Stern-Gerlach magnets, on selected components of the Spins $\boldsymbol{\sigma}_A$ and $\boldsymbol{\sigma}_B$. If measurement of the component $\mathbf{a}\cdot\boldsymbol{\sigma}_A$, where **a** is some unit vector, yields the value +1 then, according to quantum mechanics, measurement of $\mathbf{a}\cdot\boldsymbol{\sigma}_B$ must yield the value -1 and vice versa.
>
> [Now assume] that if the two measurements are made at places remote from one another the orientation of one magnet does not influence the result obtained with the other. Since we can predict in advance the result of measuring any chosen component of $\boldsymbol{\sigma}_B$, by previously measuring the same component of $\boldsymbol{\sigma}_A$, it follows that the result of any such measurement must actually be predetermined. Since the initial quantum mechanical wave function does not determine the result of an individual measurement, this predetermination implies the possibility of a more complete specification of the state.
>
> Let this more complete specification be effected by means of parameters λ. It is a matter of indifference in the following whether λ denotes a single variable or a set, or even a set of functions, and whether the variables are discrete or continuous. However, we write as if λ were a single continuous parameter.
>
> The result *A* of measuring $\mathbf{a}\cdot\boldsymbol{\sigma}_A$ is then determined by **a** and λ, and the result *B* of measuring $\mathbf{b}\cdot\boldsymbol{\sigma}_B$ in the same instance is determined by **b** and λ. If *P*(λ) is the probability distribution of λ then the expectation value of the product of the two components $\mathbf{a}\cdot\boldsymbol{\sigma}_A$ and $\mathbf{b}\cdot\boldsymbol{\sigma}_B$ is
>
> $$\int A_{\mathbf{a}}(\lambda)B_{\mathbf{b}}(\lambda)P(\lambda)\mathrm{d}\lambda$$

[1] J. S. Bell, „On the Einstein--Podolsky--Rosen paradox," Physics Physique Fizika **1**, 195-200 (1964)

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> Is quantum mechanics a classical theory in disguise?

Bell's model of classical description: "hidden" variables

$$\left[\text{"Complete specification"}\right] = \Big(\lambda \sim P(\lambda)\Big)$$

$$\left[\text{Expectation value of observable } X\right] = \lim_{n\to\infty} \frac{\sum_{i=1}^n x_i}{n} = \int X(\lambda) P(\lambda) \mathrm{d}\lambda$$

$$\left[\text{Probability that } X \text{ had value } x\right] = \int \delta_{x,X(\lambda)} P(\lambda) \mathrm{d}\lambda$$

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Bell's theorem: Indirect proof that this model doesn't fit quantum mechanics.

Meta-theorist's toolkit: Formalisms and Reformulations

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What is reformulation?

Physical Theory $\neq$ **Formalism**

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What is reformulation?

$$\left(\textbf{Physical Theory} \right) \neq \left(\textbf{Formalism} \right)$$

e.g., Classical Mechanics:

Newton Formulation	**Hamilton Formulation**
$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$	$\dot{q} = \frac{\partial \mathcal{H}(q,p)}{\partial p} \quad \dot{p} = -\frac{\partial \mathcal{H}(q,p)}{\partial q}$

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e.g., the Phenomenological method [1,2]:

$$\left(\begin{array}{c}\textbf{Inputs:}\\ \text{phenomenological}\\ \text{observations}\end{array}\right) = P^{X_n,\dots,X_1}_{t_n,\dots,t_1}(x_n,\dots,x_1) = P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n)$$

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$$\underline{z}_n = (z_n, z_{n-1}, \ldots, z_1)$$

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Inputs: phenomenological observations $= P^{X_n,\ldots,X_1}_{t_n,\ldots,t_1}(x_n,\ldots,x_1) = P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n)$ **Deduction** → **Formalism**

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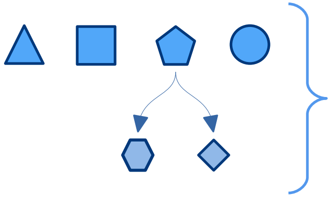
Meta-theorist's toolkit: Master Object and Interpretation

The Master Object of theory



All the 'useful' and 'needed' formulae / formal objects

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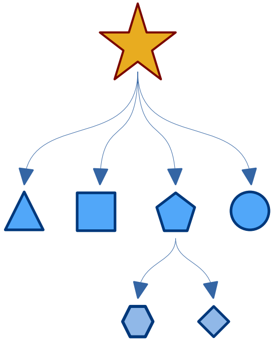

The source of all 'useful' objects, a.k.a., the **Master Object**

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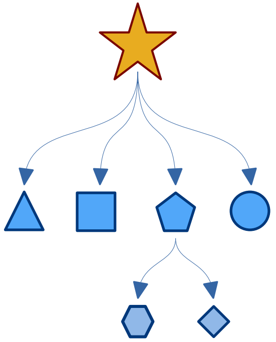

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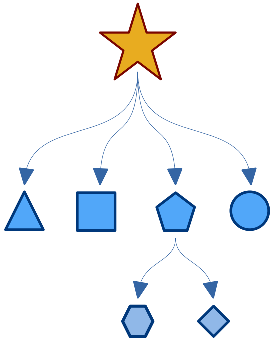

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 $= P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n)$ **Interpretation** → **User's experience:** obtaining a sequence of results

Classical theory = (Uni-)trajectory theory

Uni-trajectory formalism

$$\left(\star \textbf{ Master Object} \right) = P[\,e\,]$$

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Positive:

$$P[\,e\,] \geq 0$$

Normalized:

$$\int P[\,e\,][\mathrm{D}e] = 1$$

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(Uni-)Trajectory

$$t \mapsto e(t)$$

(With suitable generalizations, e.g.:

$$(t, \vec{r}) \mapsto \mathbf{E}(t, \vec{r})$$
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Let's elevate the Bell's idea

> The theory is **Classical** when it can be reformulated as a uni-trajectory theory.

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From **Equations of Motion** to uni-trajectory formalism:

$$P[\,e\,] \propto \mathrm{e}^{-S[\,e\,]}$$

← **Action**

Quantum theory = Bi-trajectory theory [1,2,3]

Bi-trajectory formalism

$$\left(★ \textbf{ Master Object} \right) = Q[\,\psi^{+}, \psi^{-}\,]$$

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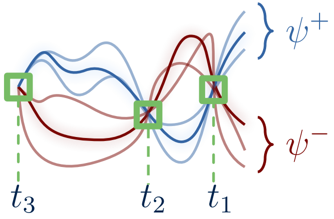

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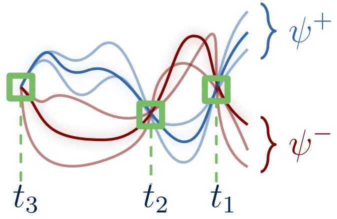

Multi-time correlations:

$$\langle [\hat{X}(t_2), \hat{Y}(t_1)] \rangle = \iint \Big\{ X\big(\psi^+(t_2, \hat{U}_x)\big) Y\big(\psi^+(t_1, \hat{U}_y)\big) - X\big(\psi^-(t_2, \hat{U}_x)\big) Y\big(\psi^-(t_1, \hat{U}_y)\big) \Big\} Q[\,\psi^+, \psi^-\,][\mathrm{D}\psi^+][\mathrm{D}\psi^-]$$

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These were the observed results because the system followed a certain trajectory:

$P[\, e \,] \to \delta[\, e - a \,]$ where $X\big(a(t_i)\big) = x_i$

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$$P[\, e \,] \to \delta[\, e - a \,] \quad \text{where } X\big(a(t_i)\big) = x_i$$

Classical interpretation of probability

$$p_a \underbrace{\delta[\, e - a \,]}_{X(a(t_i)) = x_i} + p_b \underbrace{\delta[\, e - b \,]}_{X(b(t_i)) = x_i'}$$

“Classical vibes” explained: Realism

The „realistic” explanation

Experiment:
We are given a single instance of a sequential measurement of obs. X:

$$X@\ \underline{t}_n = (t_n, \ldots, t_1)\text{:}\quad \underline{x}_n = (x_n, \ldots, x_1)$$

→

The “realistic” explanation:
These were the observed results because the system followed a certain trajectory:

$$P[\,e\,] \to \delta[\,e - a\,] \quad \text{where } X\big(a(t_i)\big) = x_i$$

Classical interpretation of probability

$$\underbrace{p_a\delta[\,e - a\,]}_{X(a(t_i)) = x_i} + \underbrace{p_b\delta[\,e - b\,]}_{X(b(t_i)) = x_i'} \xrightarrow{\text{Measurement}} \left(\underline{x}_n \text{ or } \underline{x}_n' \right)$$

“Classical vibes” explained: Realism

The „realistic” explanation

Experiment:
We are given a single instance of a sequential measurement of obs. X:

$X@\,\underline{t}_n = (t_n, \dots, t_1)$: $\underline{x}_n = (x_n, \dots, x_1)$

→

The “realistic” explanation:
These were the observed results because the system followed a certain trajectory:

$P[e] \to \delta[e-a]$ where $X(a(t_i)) = x_i$

Classical interpretation of probability

$$p_a \underbrace{\delta[e-a]}_{X(a(t_i)) = x_i} + p_b \underbrace{\delta[e-b]}_{X(b(t_i)) = x_i'}$$

Measurement →

$\underline{x}_n$ or $\underline{x}'_n$

Realistic explanation

$\underline{x}_n$ → $P[e] \to \delta[e-a]$

$\underline{x}'_n$ → $P[e] \to \delta[e-b]$

“Classical vibes” explained: Realism

The „realistic” explanation

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We are given a single instance of a sequential measurement of obs. X:

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$$p_a \underbrace{\delta[\,e - a\,]}_{X(a(t_i)) = x_i} + p_b \underbrace{\delta[\,e - b\,]}_{X(b(t_i)) = x_i'}$$

Measurement →

$\underline{x}_n$ → $P[\,e\,] \to \delta[\,e - a\,]$

or

$\underline{x}'_n$ → $P[\,e\,] \to \delta[\,e - b\,]$

Realistic explanation

The Observer “updates their priors”

“Classical vibes” explained: Realism

The „realistic” explanation

Experiment:
We are given a single instance of a sequential measurement of obs. X:

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Measurement →

$\underline{x}_n$ or $\underline{x}'_n$

Realistic explanation

$\underline{x}_n$: $P[\,e\,] \to \delta[\,e - a\,]$

$\underline{x}'_n$: $P[\,e\,] \to \delta[\,e - b\,]$

The Observer “updates their priors”

Realistic states in bi-trajectory theory?

$$Q[\,\psi^+, \psi^-\,] \to \delta[\,\psi^+ - a\,]\delta[\,\psi^- - b\,]\ \text{?}$$

"Classical vibes" explained: Realism

The „realistic" explanation

Experiment:
We are given a single instance of a sequential measurement of obs. X:

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The "realistic" explanation:
These were the observed results because the system followed a certain trajectory:

$$P[e] \rightarrow \delta[e-a] \quad \text{where } X(a(t_i)) = x_i$$

Classical interpretation of probability

$$p_a\underbrace{\delta[e-a]}_{X(a(t_i))=x_i} + p_b\underbrace{\delta[e-b]}_{X(b(t_i))=x'_i}$$

Measurement →

$\underline{x}_n$ → $P[e] \rightarrow \delta[e-a]$

or

$\underline{x}'_n$ → $P[e] \rightarrow \delta[e-b]$

Realistic explanation

The Observer "updates their priors"

Realistic states in bi-trajectory theory?

$$\left(p_a p_b \geq |r|^2\right)$$

$$Q[\psi^+, \psi^-] \rightarrow p_a\delta[\psi^+ - a]\delta[\psi^- - a] + p_b\delta[\psi^+ - b]\delta[\psi^- - b] + \underbrace{r\delta[\psi^+ - a]\delta[\psi^- - b] + r^*\delta[\psi^+ - b]\delta[\psi^- - a]}_{\text{Quantum interference!}}$$

"Classical vibes" explained: Realism

The „realistic" explanation

Experiment:
We are given a single instance of a sequential measurement of obs. X:

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Measurement →

$\underline{x}_n$ → $P[\,e\,] \to \delta[\,e - a\,]$

or — **Realistic explanation**

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The Observer "updates their priors"

Realistic states in bi-trajectory theory?

$$\left(p_a p_b \geq |r|^2\right)$$

$$Q[\,\psi^+, \psi^-\,] \to p_a\delta[\psi^+ - a]\delta[\psi^- - a] + p_b\delta[\psi^+ - b]\delta[\psi^- - b] + r\delta[\psi^+ - a]\delta[\psi^- - b] + r^*\delta[\psi^+ - b]\delta[\psi^- - a]$$

Alternatively, go to the classical limit

$$Q[\,\psi^+, \psi^-\,] \propto \delta[\,\psi^+ - \psi^-\,]$$

“Classical vibes” explained: Non-contextuality

The elementary observable: „hidden” variable doesn’t have to be hidden!

$$\begin{pmatrix}\text{Expectation value} \\ \text{of obs. } X \text{ @ } t_1\end{pmatrix} = \int X(e(t_1))P[\,e\,][\mathrm{D}e]$$

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$$\left(\begin{array}{c}\text{Expectation value of the}\\ \textbf{Elementary obs. } E \text{ @ } t_1\end{array}\right) = \int e(t_1)P[e][\mathrm{D}e] = \sum_{e_1} e_1 P^E_{t_1}(e_1)$$

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Equivalent observables: essentially, there is only one observable E

$$\left(\begin{array}{c}\text{Measurements}\\ \text{of obs. } X\end{array}\right) = P^X_{\underline{t}_n}(\underline{x}_n) = \int \Big\{\prod_{i=1}^{n}\delta_{x_i,X(e(t_i))}\Big\}P[e][De]$$

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$$\left(\begin{array}{c}\text{Expectation value of the}\\ \textbf{Elementary obs. } E \text{ @ } t_1\end{array}\right) = \int e(t_1)P[e][\mathrm{D}e] = \sum_{e_1} e_1 P^E_{t_1}(e_1)$$

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$$\left(\begin{array}{c}\text{Measurements}\\ \text{of obs. } X\end{array}\right) = P^X_{\underline{t}_n}(\underline{x}_n) = \int \Big\{\prod_{i=1}^n \underbrace{\delta_{x_i,X(e(t_i))}}_{x_i = X(e_i)}\Big\}P[e][\mathrm{D}e] = \int \Big\{\prod_{i=1}^n \underbrace{\delta_{X^{-1}(x_i),e(t_i)}}_{X^{-1}(x_i) = e_i}\Big\}P[e][\mathrm{D}e] = P^E_{\underline{t}_n}(X^{-1}(\underline{x}_n)) = \left(\begin{array}{c}\text{Measurements of the}\\ \textbf{elementary obs. } E\end{array}\right)$$

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Certainty relations: consider a rapid sequential observation

$$\lim_{\Delta t \to 0} P^{E,E}_{t+\Delta t,t}(e_2, e_1) \propto \delta_{e_2,e_1}$$

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Certainty relations: consider a rapid sequential observation

$$\lim_{\Delta t\to 0} P^{E,E}_{t+\Delta t,t}(e_2,e_1) \propto \delta_{e_2,e_1} \quad \Rightarrow \quad \lim_{\Delta t\to 0} P^{X,Y}_{t+\Delta t,t}(x,y) = \lim_{\Delta t\to 0} P^{E,E}_{t+\Delta t,t}(X^{-1}(x),Y^{-1}(y)) \propto \delta_{X^{-1}(x),Y^{-1}(y)}$$

“Classical vibes” explained: Non-contextuality

Bi-trajectory theory: inequivalent observables are possible!

$$P^{X,Y}_{t+\Delta t,t}(x,y) = \iint \left\{ \delta_{x,X(\psi^+(t+\Delta t,\hat{U}_x))} \delta_{x,X(\psi^-(t+\Delta t,\hat{U}_x))} \delta_{y,Y(\psi^+(t,\hat{U}_y))} \delta_{y,Y(\psi^-(t,\hat{U}_y))} \right\} Q[\,\psi^+,\psi^-\,][\mathrm{D}\psi^+][\mathrm{D}\psi^-]$$

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$$P^{X,Y}_{t+\Delta t,t}(x,y) = \iint \Big\{ \delta_{x,\underbrace{X(\psi^+(t+\Delta t,\hat U_x))}} \delta_{x,X(\psi^-(t+\Delta t,\hat U_x))} \delta_{y,\underbrace{Y(\psi^+(t,\hat U_y))}} \delta_{y,Y(\psi^-(t,\hat U_y))} \Big\} Q[\psi^+,\psi^-][\mathrm{D}\psi^+][\mathrm{D}\psi^-]$$

$$\hat X = \sum_{\psi=1}^{d} X(\psi)\hat U_x|\psi\rangle\langle\psi|\hat U_x^\dagger \qquad \hat Y = \sum_{\psi=1}^{d} Y(\psi)\hat U_y|\psi\rangle\langle\psi|\hat U_y^\dagger$$

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$$\hat{X} = \sum_{\psi=1}^{d} X(\psi)\hat{U}_x|\psi\rangle\langle\psi|\hat{U}_x^\dagger \qquad \hat{Y} = \sum_{\psi=1}^{d} Y(\psi)\hat{U}_y|\psi\rangle\langle\psi|\hat{U}_y^\dagger$$

Probing the same observables: classical-like behavior

$$\lim_{\Delta t\to 0} P^{X,X}_{t+\Delta t,t}(x_2,x_1) \propto \delta_{x_2,x_1} \qquad \lim_{\Delta t\to 0} P^{Y,Y}_{t+\Delta t,t}(y_2,y_1) \propto \delta_{y_2,y_1}$$

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$$P^{X,Y}_{t+\Delta t,t}(x,y) = \iint \left\{ \delta_{x,\underbrace{X(\psi^+(t+\Delta t,\hat{U}_x))}} \delta_{x,X(\psi^-(t+\Delta t,\hat{U}_x))} \delta_{y,\underbrace{Y(\psi^+(t,\hat{U}_y))}} \delta_{y,Y(\psi^-(t,\hat{U}_y))} \right\} Q[\psi^+,\psi^-][D\psi^+][D\psi^-]$$

$$\hat{X} = \sum_{\psi=1}^{d} X(\psi)\hat{U}_x|\psi\rangle\langle\psi|\hat{U}_x^\dagger \qquad \hat{Y} = \sum_{\psi=1}^{d} Y(\psi)\hat{U}_y|\psi\rangle\langle\psi|\hat{U}_y^\dagger$$

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Uncertainty relations: whole spectrum of cases!

$$\lim_{\Delta t\to 0} P^{X,Y}_{t+\Delta t,t}(x,y) = \lim_{\Delta t\to 0} P^{Y,X}_{t+\Delta t,t}(y,x) \propto \mathrm{C}^{XY}_{x,y}$$

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$$\hat{X} = \sum_{\psi=1}^{d} X(\psi)\hat{U}_x|\psi\rangle\langle\psi|\hat{U}_x^\dagger \qquad \hat{Y} = \sum_{\psi=1}^{d} Y(\psi)\hat{U}_y|\psi\rangle\langle\psi|\hat{U}_y^\dagger$$

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Commuting Observables ⟶ Mutually Unbiased Bases

$$\mathrm{C}^{XY}_{x,y} = \delta_{X^{-1}(x),Y^{-1}(y)} \qquad \mathrm{C}^{XY}_{x,y} = \frac{1}{d}$$

“Classical vibes” explained: Non-invasiveness

Mathematics of trajectories: the Consistency conditions

Uni-trajectory

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \int \left\{ \prod_{k=1}^{n} \delta_{x_k, X(e(t_k))} \right\} P[\,e\,][\mathrm{D}e]$$

Trivial math...

$$\sum_{x_i} P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = P^{X_n,\dots,\cancel{X_i},\dots,X_1}_{t_n,\dots,\cancel{t_i},\dots,t_1}(x_n,\dots,\cancel{x_i},\dots,x_1)$$

“Classical vibes” explained: Non-invasiveness

Mathematics of trajectories: the Consistency conditions

Uni-trajectory

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \int \left\{ \prod_{k=1}^{n} \delta_{x_k, X(e(t_k))} \right\} P[\,e\,][\mathrm{D}e]$$

Trivial math...

Extension Theorem

(non-trivial math)

$$\sum_{x_i} P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = P^{X_n,\ldots,\cancel{X_i},\ldots,X_1}_{t_n,\ldots,\cancel{t_i},\ldots,t_1}(x_n,\ldots,\cancel{x_i},\ldots,x_1)$$

“Classical vibes” explained: Non-invasiveness

Mathematics of trajectories: the Consistency conditions

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$$P^{X_n}_{\underline{t}_n}(\underline{x}_n) = \int \left\{ \prod_{k=1}^{n} \delta_{x_k, X(e(t_k))} \right\} P[\,e\,][\mathrm{D}e]$$

Trivial math... →

← **Extension Theorem** (non-trivial math)

$$\sum_{x_i} P^{X_n}_{\underline{t}_n}(\underline{x}_n) = P^{X_n,\dots,\cancel{X_i},\dots,X_1}_{t_n,\dots,\cancel{t_i},\dots,t_1}(x_n,\dots,\cancel{x_i},\dots,x_1)$$

Bi-trajectory

$$Q^{X_n}_{\underline{t}_n}(\underline{x}_n^+;\underline{x}_n^-) = \iint \left\{ \prod_{k=1}^{n} \delta_{x_k^+, X(\psi^+(t_k,\hat{U}_k))} \delta_{x_k^-, X(\psi^-(t_k,\hat{U}_k))} \right\} Q[\,\psi^+,\psi^-\,][\mathrm{D}\psi^+][\mathrm{D}\psi^-]$$

⇄ [1]

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[1] D.Lonigro, F.Sakuldee, Ł.Cywiński, D.Chruściński, P.S., „Double or nothing: a Kolmogorov extension theorem for multitime (bi)probabilities in quantum mechanics,” Quantum 8, 1447 (2024)

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Mathematics of trajectories: the Consistency conditions

Uni-trajectory

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \int \left\{\prod_{k=1}^{n} \delta_{x_k, X(e(t_k))}\right\} P[\,e\,][De]$$

Trivial math... →

← **Extension Theorem** (non-trivial math)

$$\sum_{x_i} P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = P^{X_n,\dots,\cancel{X_i},\dots,X_1}_{t_n,\dots,\cancel{t_i},\dots,t_1}(x_n,\dots,\cancel{x_i},\dots,x_1)$$

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Physical consequences: Quantum mechanics is not a uni-trajectory theory (duh!)

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Where is the Non-invasiveness?

$$P^{\cdots\cancel{X_i}\cdots}_{\cdots\cancel{t_i}\cdots}(\ldots\cancel{x_i}\ldots) = \left(\begin{array}{c}\text{Measuring device} \\ \textbf{not}\text{ deployed @ } t_i\end{array}\right)$$

$$\sum_{x_i} P^{\ldots X_i\ldots}_{\ldots t_i\ldots}(\ldots x_i\ldots) = \left(\begin{array}{c}\text{Device deployed @ } t_i \\ \text{but ignored / forgotten}\end{array}\right)$$

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$$Q^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n;\underline{x}_n) = P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n)$$

→

$$\sum_{x_i} P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = P^{\ldots\cancel{X_i}\ldots}_{\ldots\cancel{t_i}\ldots}(\ldots\cancel{x_i}\ldots) - \underbrace{\sum_{x_i^+\neq x_i^-} Q^{\ldots X_i\ldots}_{\ldots t_i\ldots}(\ldots x_i^+\ldots;\ldots x_i^-\ldots)}_{\text{Quantum interference!}}$$

Where is the Non-invasiveness?

$$P^{\ldots\cancel{X_i}\ldots}_{\ldots\cancel{t_i}\ldots}(\ldots\cancel{x_i}\ldots) =$$ (Measuring device **not deployed @** t_i)

$$\sum_{x_i} P^{\ldots X_i\ldots}_{\ldots t_i\ldots}(\ldots x_i\ldots) =$$ (Device deployed @ t_i but ignored / forgotten)

\+ (**Assume Non-invasiveness:** Measuring devices don’t affect the measured system) → (**Consistency condition**)

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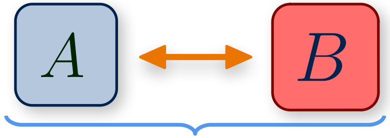
“Classical vibes” explained: Non-invasiveness

Interacting systems, in general

Composite system AB:



$P_{AB}[\,a \wedge b\,]$

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Interacting systems, in general

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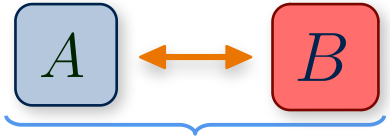

$P_{AB}[\,a \wedge b\,]$

Subsystem A (conditioned by B):

$$P_{A|B}[\,a\,] = \int P_{AB}[\,a \wedge b\,][\mathrm{D}b]$$

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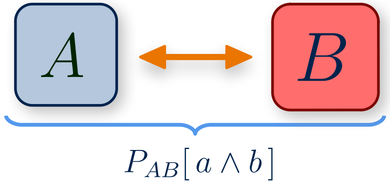

Subsystem A (conditioned by B):

$$P_{A|B}[\,a\,] = \int P_{AB}[\,a \wedge b\,][\mathrm{D}b] = \int K[\,a \mid b\,]P_B[\,b\,][\mathrm{D}b]$$

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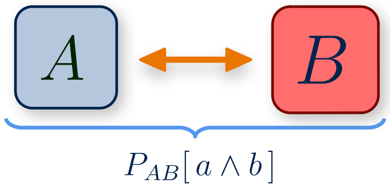

Subsystem A (conditioned by B):

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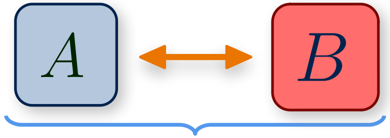

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From interacting systems to measuring device

$$\int K[\,a \mid b\,]P_B[\,b\,][\mathrm{D}b] \xrightarrow{K \text{ is a lossless filter}} P_{A|B}[\,a\,] \to P_B[\,b\,] \quad \left(\int \left\{\prod_i \delta_{x_i, X(b(t_i))}\right\} P_B[\,b\,][\mathrm{D}b]\right)$$

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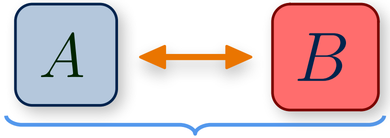

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$$\int K[\,a \mid b\,]P_B[\,b\,][\mathrm{D}b]$$

K is a lossless filter →

$$P_{A|B}[\,a\,] \to P_B[\,b\,] \quad \left(\int \Big\{\prod_i \delta_{x_i, X(b(t_i))}\Big\} P_B[\,b\,][\mathrm{D}b]\right)$$

How to quantify Non-invasiveness?

$$P_{B|A}[\,b\,] = \int G[\,b \mid a\,]P_A[\,a\,][\mathrm{D}a]$$

“Classical vibes” explained: Non-invasiveness

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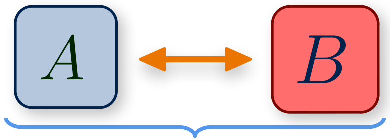

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$$\int K[\,a \mid b\,]P_B[\,b\,][\mathrm{D}b] \xrightarrow{K \text{ is a lossless filter}} P_{A|B}[\,a\,] \to P_B[\,b\,] \quad \left(\int \Big\{\prod_i \delta_{x_i,X(b(t_i))}\Big\} P_B[\,b\,][\mathrm{D}b]\right)$$

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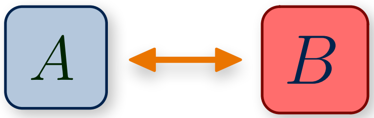

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From interacting systems to measuring device

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Quantum Mechanics

$$Q_{A|B}[\alpha^+, \alpha^-] = \iint K[\alpha^+, \alpha^- \mid \beta^+, \beta^-] Q_B[\beta^+, \beta^-][\mathrm{D}\beta^+][\mathrm{D}\beta^-]$$

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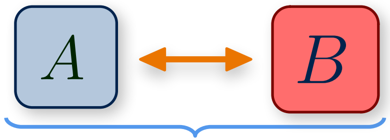

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Quantum Mechanics

$$Q_{A|B}[\,\alpha^+,\alpha^-\,] = \iint K[\,\alpha^+,\alpha^- \mid \beta^+,\beta^-\,]Q_B[\,\beta^+,\beta^-\,][\mathrm{D}\beta^+][\mathrm{D}\beta^-]$$

Extra step:
Measuring device must be classical!

$$\longrightarrow Q_{A|B}[\,\alpha^+,\alpha^-\,] \to \delta[\alpha^+ - \alpha^-]P_{A|B}[\,\alpha^+\,]$$

New perspective on Bell's question

> Is quantum mechanics a classical theory in disguise?

Why Bell inequalities?

New perspective on Bell's question

> Is quantum mechanics a classical theory in disguise?

Why Bell inequalities?

$$\mathrm{Corr}(\mathbf{a},\mathbf{b}) = \int A_{\mathbf{a}}(\lambda) B_{\mathbf{b}}(\lambda) \Big(\int \delta(e(0)-\lambda) P[\,e\,][\mathrm{D}e] \Big) \mathrm{d}\lambda$$

New perspective on Bell's question

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Why Bell inequalities?

$$\mathrm{Corr}(\mathbf{a},\mathbf{b}) = \int \underbrace{A_{\mathbf{a}}(\lambda)B_{\mathbf{b}}(\lambda)}_{A_{\mathbf{a}},\, B_{\mathbf{b}} \,:\, \mathbb{R} \to \{-1,1\}} \left(\int \delta(e(0)-\lambda) P[\,e\,][\mathrm{D}e] \right) \mathrm{d}\lambda$$

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[1] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt, "Proposed experiment to test local hidden-variable theories", Phys. Rev. Lett. 23, 880-4 (1969)

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$$\int A_{\mathbf{a}}(\lambda)B_{\mathbf{b}}(\lambda)P(\lambda)\mathrm{d}\lambda \;\longleftrightarrow\; \langle \hat{A}_{\mathbf{a}} \otimes \hat{B}_{\mathbf{b}} \rangle$$

(Theory) $\neq$ (Formalism)

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$$\int A_{\mathbf{a}}(\lambda) B_{\mathbf{b}}(\lambda) P(\lambda) \mathrm{d}\lambda \;\longleftrightarrow\; \langle \underbrace{\hat{A}_{\mathbf{a}} \otimes \hat{B}_{\mathbf{b}}}\rangle$$

(Theory) $\neq$ (Formalism)

$\hat{A}_{\mathbf{a}} = \mathbf{a} \cdot \hat{\boldsymbol{\sigma}}_A \quad \hat{B}_{\mathbf{b}} = \mathbf{b} \cdot \hat{\boldsymbol{\sigma}}_B$

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$$\int A_{\mathbf{a}}(\lambda)B_{\mathbf{b}}(\lambda)P(\lambda)\mathrm{d}\lambda \quad \longleftrightarrow \quad \langle \hat{A}_{\mathbf{a}} \otimes \hat{B}_{\mathbf{b}} \rangle$$

(Theory) ≠ (Formalism)

$$\big|\langle \hat{A}_{\mathbf{u}} \otimes \hat{B}_{\mathbf{v}} \rangle - \langle \hat{A}_{\mathbf{u}} \otimes \hat{B}_{\mathbf{v}'} \rangle + \langle \hat{A}_{\mathbf{u}'} \otimes \hat{B}_{\mathbf{v}} \rangle + \langle \hat{A}_{\mathbf{u}'} \otimes \hat{B}_{\mathbf{v}'} \rangle\big| = 2\sqrt{2} > 2$$

[1] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt, "Proposed experiment to test local hidden-variable theories", Phys. Rev. Lett. 23, 880-4 (1969)

New perspective on Bell's question

> Is quantum mechanics a classical theory in disguise?

Why Bell inequalities?

$$\mathrm{Corr}(\mathbf{a},\mathbf{b}) = \int A_{\mathbf{a}}(\lambda)B_{\mathbf{b}}(\lambda)\Big(\int \delta(e(0)-\lambda)P[\,e\,][\mathrm{D}e]\Big)\mathrm{d}\lambda \quad\Longrightarrow\quad \big|\mathrm{Corr}(\mathbf{a},\mathbf{b}) - \mathrm{Corr}(\mathbf{a},\mathbf{b}') + \mathrm{Corr}(\mathbf{a}',\mathbf{b}) + \mathrm{Corr}(\mathbf{a}',\mathbf{b}')\big| \leq 2$$ [1]

(Theory) ≠ (Formalism)

$$\int A_{\mathbf{a}}(\lambda)B_{\mathbf{b}}(\lambda)P(\lambda)\mathrm{d}\lambda \quad\Longleftrightarrow\quad \langle \hat{A}_{\mathbf{a}} \otimes \hat{B}_{\mathbf{b}} \rangle$$

$$\big|\langle \hat{A}_{\mathbf{u}} \otimes \hat{B}_{\mathbf{v}} \rangle - \langle \hat{A}_{\mathbf{u}} \otimes \hat{B}_{\mathbf{v}'} \rangle + \langle \hat{A}_{\mathbf{u}'} \otimes \hat{B}_{\mathbf{v}} \rangle + \langle \hat{A}_{\mathbf{u}'} \otimes \hat{B}_{\mathbf{v}'} \rangle\big| = 2\sqrt{2} > 2$$

The direct answer

> **Question**:
> Is quantum mechanics a uni-trajectory theory?

→

> **Answer**:
> No, because it is a bi-trajectory theory.

[1] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt, "Proposed experiment to test local hidden-variable theories", Phys. Rev. Lett. 23, 880-4 (1969)

End

[1] P.S., D.Lonigro, F.Sakuldee, Ł.Cywiński, D.Chruściński, „Phenomenological quantum mechanics I: phenomenology of quantum observables," arXiv 2410.14410
[2] P.S., D.Lonigro, F.Sakuldee, Ł.Cywiński, D.Chruściński, „Phenomenological quantum mechanics II: deducing the formalism from experimental observations," arXiv 2507.04812
[3] D.Lonigro, F.Sakuldee, Ł.Cywiński, D.Chruściński, P.S., „Double or nothing: a Kolmogorov extension theorem for multitime (bi)probabilities in quantum mechanics," Quantum **8**, 1447 (2024)
[4] P.S., Ł.Cywinski, „Objectivity of classical quantum stochastic processes," Quantum **8**, 1390 (2024)
[5] P.S., „Foundations of Quantum Theory," youtube.com/@theory_ifpan, http://www.theory.ifpan.edu.pl/?class=24/1/foundqm/pszan

Appendix A: the Standard Formalism

The standard formalism: a dichotomous mode of description

Measurement Context

The Heisenberg cut

Non-measurement Context

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State:

$$\hat{\rho}_t = \hat{U}_{t,t_0}\hat{\rho}_{t_0}\hat{U}_{t_0,t}$$

$$\hat{U}_{t,t_0} = \mathrm{e}^{-\mathrm{i}(t-t_0)\hat{H}}$$

$$\hat{P}^X_t(x) := \sum_{\psi=1}^{d} \delta_{x,X(\psi)}\hat{U}_{0,t}\hat{U}_x|\psi\rangle\langle\psi|\hat{U}^\dagger_x\hat{U}_{t,0}$$

← NInv Bi

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Born rule:

$$P^X(x \mid \rho_t) = \mathrm{tr}\left[\hat{P}_0^X(x)\hat{\rho}_t\right] = \mathrm{tr}\left[\hat{P}_t^X(x)\hat{\rho}_{t_0}\hat{P}_t^X(x)\right]$$

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$$P_{\underline{t}_n}^{\underline{X}_n}(\underline{x}_n) = P^{X_1}(x_1 \mid \rho_{t_1}) \prod_{k=1}^{n-1} P^{X_{k+1}}(x_{k+1} \mid \rho_{t_{k+1}|\underline{t}_k}(\underline{x}_k))$$

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The Heisenberg cut

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$$\hat{\rho}_t = \hat{U}_{t,t_0}\hat{\rho}_{t_0}\hat{U}_{t_0,t}$$

Master Object?

Collapsed state:

$$\hat{\rho}_{t|\underline{t}_n}(\underline{x}_n) = \hat{U}_{t,t_n} \frac{\hat{P}_0^{X_n}(x_n)\hat{\rho}_{t_n|\underline{t}_{n-1}}(\underline{x}_{n-1})\hat{P}_0^{X_n}(x_n)}{P^{X_n}\left(x_n \mid \rho_{t_n|\underline{t}_{n-1}}(\underline{x}_{n-1})\right)} \hat{U}_{t_n,t}$$

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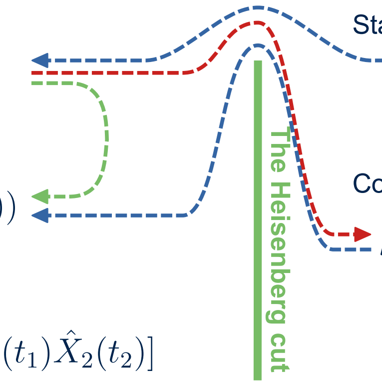

Non-measurement Context

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The Heisenberg cut

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$$\hat{\rho}_t = \hat{U}_{t,t_0}\hat{\rho}_{t_0}\hat{U}_{t_0,t}$$

Master Object?

Collapsed state:

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Multi-time correlations:

$$\left\langle[\hat{X}_2(t_2),\hat{X}_1(t_1)]\right\rangle = \mathrm{tr}\left[\hat{X}(t_2)\hat{X}(t_1)\hat{\rho}_{t_0}\right] - \mathrm{tr}\left[\hat{\rho}_{t_0}\hat{X}_1(t_1)\hat{X}_2(t_2)\right]$$
$$= \sum_{\underline{x}_2^{\pm}}(x_2^+x_1^+ - x_2^-x_1^-)$$
$$\times\, \mathrm{tr}\left[\hat{P}_{t_2}^{X_2}(x_2^+)\hat{P}_{t_1}^{X_1}(x_1^+)\hat{\rho}_{t_0}\hat{P}_{t_1}^{X_1}(x_1^-)\hat{P}_{t_2}^{X_2}(x_2^-)\right]$$

The Heisenberg cut

Non-measurement Context

State:

$$\hat{\rho}_t = \hat{U}_{t,t_0}\hat{\rho}_{t_0}\hat{U}_{t_0,t}$$

← (⭐ **Master Object?**)

Collapsed state:

$$\hat{\rho}_{t|\underline{t}_n}(\underline{x}_n) = \hat{U}_{t,t_n}\frac{\hat{P}_0^{X_n}(x_n)\hat{\rho}_{t_n|\underline{t}_{n-1}}(\underline{x}_{n-1})\hat{P}_0^{X_n}(x_n)}{P^{X_n}\left(x_n \mid \rho_{t_n|\underline{t}_{n-1}}(\underline{x}_{n-1})\right)}\hat{U}_{t_n,t}$$

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← NInv Bi

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Measurement Context v2

Born rule v2:

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \mathrm{tr}\left[\hat{P}^{X_n}_{t_n}(x_n)\cdots\hat{P}^{X_1}_{t_1}(x_1)\hat{\rho}_{t_0}\hat{P}^{X_1}_{t_1}(x_1)\cdots\hat{P}^{X_n}_{t_n}(x_n)\right]$$

Multi-time correlations:

$$\langle[\hat{X}_2(t_2),\hat{X}_1(t_1)]\rangle = \mathrm{tr}\left[\hat{X}(t_2)\hat{X}(t_1)\hat{\rho}_{t_0}\right] - \mathrm{tr}\left[\hat{\rho}_{t_0}\hat{X}_1(t_1)\hat{X}_2(t_2)\right]$$

$$= \sum_{\underline{x}_2^{\pm}}(x_2^+x_1^+ - x_2^-x_1^-)$$

$$\times\, \mathrm{tr}\left[\hat{P}^{X_2}_{t_2}(x_2^+)\hat{P}^{X_1}_{t_1}(x_1^+)\hat{\rho}_{t_0}\hat{P}^{X_1}_{t_1}(x_1^-)\hat{P}^{X_2}_{t_2}(x_2^-)\right]$$

The Heisenberg cut

Non-measurement Context

State:

$$\hat{\rho}_t = \hat{U}_{t,t_0}\hat{\rho}_{t_0}\hat{U}_{t_0,t}$$

Master Object?

Collapsed state:

$$\hat{\rho}_{t|\underline{t}_n}(\underline{x}_n) = \hat{U}_{t,t_n}\frac{\hat{P}^{X_n}_0(x_n)\hat{\rho}_{t_n|\underline{t}_{n-1}}(\underline{x}_{n-1})\hat{P}^{X_n}_0(x_n)}{P^{X_n}\left(x_n \mid \rho_{t_n|\underline{t}_{n-1}}(\underline{x}_{n-1})\right)}\hat{U}_{t_n,t}$$

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Measurement Context v2

Born rule v2:

$$P_{\underline{t}_n}^{\underline{X}_n}(\underline{x}_n) = \mathrm{tr}\left[\hat{P}_{t_n}^{X_n}(x_n)\cdots\hat{P}_{t_1}^{X_1}(x_1)\hat{\rho}_{t_0}\hat{P}_{t_1}^{X_1}(x_1)\cdots\hat{P}_{t_n}^{X_n}(x_n)\right]$$

Multi-time correlations:

$$\left\langle[\hat{X}_2(t_2),\hat{X}_1(t_1)]\right\rangle = \mathrm{tr}\left[\hat{X}(t_2)\hat{X}(t_1)\hat{\rho}_{t_0}\right] - \mathrm{tr}\left[\hat{\rho}_{t_0}\hat{X}_1(t_1)\hat{X}_2(t_2)\right]$$

$$= \sum_{\underline{x}_2^{\pm}}(x_2^+x_1^+ - x_2^-x_1^-) \times \mathrm{tr}\left[\hat{P}_{t_2}^{X_2}(x_2^+)\hat{P}_{t_1}^{X_1}(x_1^+)\hat{\rho}_{t_0}\hat{P}_{t_1}^{X_1}(x_1^-)\hat{P}_{t_2}^{X_2}(x_2^-)\right]$$

The Heisenberg cut

Non-measurement Context v2

State: $\hat{\rho}_t = \hat{U}_{t,t_0}\hat{\rho}_{t_0}\hat{U}_{t_0,t}$

$$\hat{U}_{t,t_0} = \mathrm{e}^{-\mathrm{i}(t-t_0)\hat{H}}$$

$$\hat{P}_t^X(x) := \sum_{\psi=1}^{d}\delta_{x,X(\psi)}\hat{U}_{0,t}\hat{U}_x|\psi\rangle\langle\psi|\hat{U}_x^\dagger\hat{U}_{t,0}$$

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Measurement Context v2

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$$P_{\underline{t}_n}^{\underline{X}_n}(\underline{x}_n) = \mathrm{tr}\left[\hat{P}_{t_n}^{X_n}(x_n)\cdots\hat{P}_{t_1}^{X_1}(x_1)\hat{\rho}_{t_0}\hat{P}_{t_1}^{X_1}(x_1)\cdots\hat{P}_{t_n}^{X_n}(x_n)\right]$$

Multi-time correlations:

$$\left\langle[\hat{X}_2(t_2),\hat{X}_1(t_1)]\right\rangle = \mathrm{tr}\left[\hat{X}(t_2)\hat{X}(t_1)\hat{\rho}_{t_0}\right] - \mathrm{tr}\left[\hat{\rho}_{t_0}\hat{X}_1(t_1)\hat{X}_2(t_2)\right]$$

$$= \sum_{\underline{x}_2^{\pm}}(x_2^+x_1^+ - x_2^-x_1^-) \times \mathrm{tr}\left[\hat{P}_{t_2}^{X_2}(x_2^+)\hat{P}_{t_1}^{X_1}(x_1^+)\hat{\rho}_{t_0}\hat{P}_{t_1}^{X_1}(x_1^-)\hat{P}_{t_2}^{X_2}(x_2^-)\right]$$

The Heisenberg cut

Non-measurement Context v2

Initial State:

$$\hat{\rho}_{t_0}$$

$$\hat{U}_{t,t_0} = \mathrm{e}^{-\mathrm{i}(t-t_0)\hat{H}}$$

$$\hat{P}_t^X(x) := \sum_{\psi=1}^{d}\delta_{x,X(\psi)}\hat{U}_{0,t}\hat{U}_x|\psi\rangle\langle\psi|\hat{U}_x^\dagger\hat{U}_{t,0}$$

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The standard formalism: a dichotomous mode of description

Measurement Context v3

Initial condition:

$$\hat{\rho}_{t_0}$$

Born rule v2:

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \mathrm{tr}\left[\hat{P}^{X_n}_{t_n}(x_n)\cdots\hat{P}^{X_1}_{t_1}(x_1)\hat{\rho}_{t_0}\hat{P}^{X_1}_{t_1}(x_1)\cdots\hat{P}^{X_n}_{t_n}(x_n)\right]$$

Multi-time correlations:

$$\left\langle[\hat{X}_2(t_2),\hat{X}_1(t_1)]\right\rangle = \mathrm{tr}\left[\hat{X}(t_2)\hat{X}(t_1)\hat{\rho}_{t_0}\right] - \mathrm{tr}\left[\hat{\rho}_{t_0}\hat{X}_1(t_1)\hat{X}_2(t_2)\right]$$
$$= \sum_{\underline{x}_2^{\pm}}(x_2^+x_1^+ - x_2^-x_1^-)\times\mathrm{tr}\left[\hat{P}^{X_2}_{t_2}(x_2^+)\hat{P}^{X_1}_{t_1}(x_1^+)\hat{\rho}_{t_0}\hat{P}^{X_1}_{t_1}(x_1^-)\hat{P}^{X_2}_{t_2}(x_2^-)\right]$$

The Heisenberg cut

Non-measurement Context v3

$$\left(\begin{aligned}&\hat{U}_{t,t_0} = \mathrm{e}^{-\mathrm{i}(t-t_0)\hat{H}}\\ &\hat{P}^X_t(x) := \sum_{\psi=1}^{d}\delta_{x,X(\psi)}\hat{U}_{0,t}\hat{U}_x|\psi\rangle\langle\psi|\hat{U}_x^\dagger\hat{U}_{t,0}\end{aligned}\right)$$

← NInv Bi

Appendix A: the Standard Formalism

The standard formalism: a dichotomous mode of description

Measurement Context v3

Non-measurement Context v3

Initial condition:

$$\hat{\rho}_{t_0} = \sum_{\psi=1}^{d} \rho(\psi)\hat{U}_0|\psi\rangle\langle\psi|\hat{U}_0^\dagger = \sum_{x_0} p(x_0)\hat{P}_{t_0}^{X_0}(x_0) = \sum_{x_0} p(x_0)\hat{P}_{t_0}^{X_0}(x_0)\hat{P}_{t_0}^{X_0}(x_0)$$

Born rule v2:

$$P_{\underline{t}_n}^{\underline{X}_n}(\underline{x}_n) = \mathrm{tr}\left[\hat{P}_{t_n}^{X_n}(x_n)\cdots\hat{P}_{t_1}^{X_1}(x_1)\hat{\rho}_{t_0}\hat{P}_{t_1}^{X_1}(x_1)\cdots\hat{P}_{t_n}^{X_n}(x_n)\right]$$

Multi-time correlations:

$$\left\langle[\hat{X}_2(t_2),\hat{X}_1(t_1)]\right\rangle = \mathrm{tr}\left[\hat{X}(t_2)\hat{X}(t_1)\hat{\rho}_{t_0}\right] - \mathrm{tr}\left[\hat{\rho}_{t_0}\hat{X}_1(t_1)\hat{X}_2(t_2)\right]$$

$$= \sum_{\underline{x}_2^{\pm}}(x_2^+x_1^+ - x_2^-x_1^-)\times\mathrm{tr}\left[\hat{P}_{t_2}^{X_2}(x_2^+)\hat{P}_{t_1}^{X_1}(x_1^+)\hat{\rho}_{t_0}\hat{P}_{t_1}^{X_1}(x_1^-)\hat{P}_{t_2}^{X_2}(x_2^-)\right]$$

The Heisenberg cut

$$\hat{U}_{t,t_0} = \mathrm{e}^{-\mathrm{i}(t-t_0)\hat{H}}$$

$$\hat{P}_t^X(x) := \sum_{\psi=1}^{d}\delta_{x,X(\psi)}\hat{U}_{0,t}\hat{U}_x|\psi\rangle\langle\psi|\hat{U}_x^\dagger\hat{U}_{t,0}$$

← NInv Bi

Appendix A: the Standard Formalism

The standard formalism: a dichotomous mode of description

Measurement Context v3

Non-measurement Context v3

Born rule v3:

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \sum_{x_0} P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0)p(x_0)$$

$$P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0) = \mathrm{tr}\left[\left(\prod_{k=n}^{0} \hat{P}^{X_k}_{t_k}(x_k)\right)\left(\prod_{k=0}^{n} \hat{P}^{X_k}_{t_k}(x_k)\right)\right]$$

Multi-time correlations:

$$\left\langle[\hat{X}_2(t_2), \hat{X}_1(t_1)]\right\rangle = \sum_{\underline{x}_2^{\pm}, x_0^{\pm}} (x_2^+ x_1^+ - x_2^- x_1^-)\, \mathrm{tr}\left[\left(\prod_{k=2}^{0} \hat{P}^{X_k}_{t_k}(x_k^+)\right)\left(\prod_{k=0}^{2} \hat{P}^{X_k}_{t_k}(x_k^-)\right)\right]\sqrt{p(x_0^+)p(x_0^-)}$$

The Heisenberg cut

$$\left(\begin{aligned} \hat{U}_{t,t_0} &= \mathrm{e}^{-\mathrm{i}(t-t_0)\hat{H}} \\ \hat{P}^X_t(x) &:= \sum_{\psi=1}^{d} \delta_{x,X(\psi)} \hat{U}_{0,t}\hat{U}_x|\psi\rangle\langle\psi|\hat{U}_x^\dagger \hat{U}_{t,0} \end{aligned}\right)$$

← NInv Bi

Appendix A: the Standard Formalism

The standard formalism: a dichotomous mode of description

Measurement Context v4

Non-measurement Context v3

Bi-probability distributions:

$$Q^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n^+;\underline{x}_n^- \mid x_0^+;x_0^-) = \mathrm{tr}\left[\left(\prod_{k=n}^{0}\hat{P}^{X_k}_{t_k}(x_k^+)\right)\left(\prod_{k=0}^{n}\hat{P}^{X_k}_{t_k}(x_k^-)\right)\right]$$

Born rule v4:

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \sum_{x_0} Q^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n;\underline{x}_n \mid x_0;x_0)p(x_0)$$

Multi-time correlations:

$$\left\langle[\hat{X}_2(t_2),\hat{X}_1(t_1)]\right\rangle = \sum_{\underline{x}_2^\pm,x_0^\pm}(x_2^+x_1^+ - x_2^-x_1^-)Q^{\underline{X}_2|X_0}_{\underline{t}_2|t_0}(\underline{x}_2^+;\underline{x}_2^- \mid x_0^+;x_0^-)\sqrt{p(x_0^+)p(x_0^-)}$$

The Heisenberg cut

$$\begin{pmatrix} \hat{U}_{t,t_0} = \mathrm{e}^{-\mathrm{i}(t-t_0)\hat{H}} \\ \hat{P}^X_t(x) := \sum_{\psi=1}^{d}\delta_{x,X(\psi)}\hat{U}_{0,t}\hat{U}_x|\psi\rangle\langle\psi|\hat{U}_x^\dagger\hat{U}_{t,0} \end{pmatrix}$$

← NInv Bi

Appendix A: the Standard Formalism

The standard formalism: a dichotomous mode of description

~~Measurement~~ All Contexts

Bi-probability distributions:

$$Q^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n^+;\underline{x}_n^- \mid x_0^+;x_0^-) = \mathrm{tr}\left[\left(\prod_{k=n}^{0}\hat{P}^{X_k}_{t_k}(x_k^+)\right)\left(\prod_{k=0}^{n}\hat{P}^{X_k}_{t_k}(x_k^-)\right)\right]$$

Born rule v4:

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \sum_{x_0} Q^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n;\underline{x}_n \mid x_0;x_0)p(x_0)$$

Multi-time correlations:

$$\left\langle[\hat{X}_2(t_2),\hat{X}_1(t_1)]\right\rangle = \sum_{\underline{x}_2^\pm,x_0^\pm}(x_2^+x_1^+ - x_2^-x_1^-)Q^{\underline{X}_2|X_0}_{\underline{t}_2|t_0}(\underline{x}_2^+;\underline{x}_2^- \mid x_0^+;x_0^-)\sqrt{p(x_0^+)p(x_0^-)}$$

$$\begin{pmatrix} \hat{U}_{t,t_0} = \mathrm{e}^{-\mathrm{i}(t-t_0)\hat{H}} \\ \hat{P}^X_t(x) := \sum_{\psi=1}^{d}\delta_{x,X(\psi)}\hat{U}_{0,t}\hat{U}_x|\psi\rangle\langle\psi|\hat{U}^\dagger_x\hat{U}_{t,0} \end{pmatrix}$$

← NInv Bi

Appendix A: the Standard Formalism

The standard formalism: a dichotomous mode of description

~~Measurement~~ All Contexts

Bi-probability distributions:

$$Q^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n^+;\underline{x}_n^- \mid x_0^+;x_0^-) = \mathrm{tr}\left[\left(\prod_{k=n}^{0}\hat{P}^{X_k}_{t_k}(x_k^+)\right)\left(\prod_{k=0}^{n}\hat{P}^{X_k}_{t_k}(x_k^-)\right)\right]$$

(☆ **Master Object**) $=$ **?**

Born rule v4:

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \sum_{x_0} Q^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n;\underline{x}_n \mid x_0;x_0)\,p(x_0)$$

Multi-time correlations:

$$\left\langle[\hat{X}_2(t_2),\hat{X}_1(t_1)]\right\rangle = \sum_{\underline{x}_2^\pm,x_0^\pm}(x_2^+x_1^+ - x_2^-x_1^-)\,Q^{\underline{X}_2|X_0}_{\underline{t}_2|t_0}(\underline{x}_2^+;\underline{x}_2^- \mid x_0^+;x_0^-)\sqrt{p(x_0^+)p(x_0^-)}$$

$$\hat{U}_{t,t_0} = \mathrm{e}^{-\mathrm{i}(t-t_0)\hat{H}}$$

$$\hat{P}^X_t(x) := \sum_{\psi=1}^{d}\delta_{x,X(\psi)}\hat{U}_{0,t}\hat{U}_x|\psi\rangle\langle\psi|\hat{U}_x^\dagger\hat{U}_{t,0}$$

← NInv Bi

Appendix A: the Standard Formalism

The standard formalism: a dichotomous mode of description

~~Measurement~~ All Contexts

Bi-probability distributions:

$$Q^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n^+;\underline{x}_n^- \mid x_0^+;x_0^-) = \mathrm{tr}\left[\left(\prod_{k=n}^{0}\hat{P}^{X_k}_{t_k}(x_k^+)\right)\left(\prod_{k=0}^{n}\hat{P}^{X_k}_{t_k}(x_k^-)\right)\right]$$

⭐ **Master Object** $= Q[\,\psi^+, \psi^-\,]$

Born rule v4:

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \sum_{x_0} Q^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n;\underline{x}_n \mid x_0;x_0)\,p(x_0)$$

Multi-time correlations:

$$\left\langle[\hat{X}_2(t_2),\hat{X}_1(t_1)]\right\rangle = \sum_{\underline{x}_2^\pm, x_0^\pm}(x_2^+x_1^+ - x_2^-x_1^-)\,Q^{\underline{X}_2|X_0}_{\underline{t}_2|t_0}(\underline{x}_2^+;\underline{x}_2^- \mid x_0^+;x_0^-)\sqrt{p(x_0^+)p(x_0^-)}$$

$$\hat{U}_{t,t_0} = \mathrm{e}^{-\mathrm{i}(t-t_0)\hat{H}}$$

$$\hat{P}^X_t(x) := \sum_{\psi=1}^{d}\delta_{x,X(\psi)}\hat{U}_{0,t}\hat{U}_x|\psi\rangle\langle\psi|\hat{U}_x^\dagger\hat{U}_{t,0}$$

← NInv Bi

Appendix A: the Standard Formalism

The ~~standard~~ bi-trajectory formalism: a ~~dichotomous~~ single mode of description

~~Measurement~~ All Contexts

Bi-probability distributions:

$$Q^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n^+;\underline{x}_n^- \mid x_0^+;x_0^-) = \mathrm{tr}\left[\left(\prod_{k=n}^{0}\hat{P}^{X_k}_{t_k}(x_k^+)\right)\left(\prod_{k=0}^{n}\hat{P}^{X_k}_{t_k}(x_k^-)\right)\right]$$

[★ **Master Object**] $= Q[\,\psi^+,\psi^-\,]$

Born rule v4:

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \sum_{x_0} Q^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n;\underline{x}_n \mid x_0;x_0)\,p(x_0)$$

Multi-time correlations:

$$\left\langle[\hat{X}_2(t_2),\hat{X}_1(t_1)]\right\rangle = \sum_{\underline{x}_2^{\pm},x_0^{\pm}}(x_2^+x_1^+ - x_2^-x_1^-)\,Q^{\underline{X}_2|X_0}_{\underline{t}_2|t_0}(\underline{x}_2^+;\underline{x}_2^- \mid x_0^+;x_0^-)\sqrt{p(x_0^+)p(x_0^-)}$$

$$\hat{U}_{t,t_0} = \mathrm{e}^{-\mathrm{i}(t-t_0)\hat{H}}$$

$$\hat{P}^X_t(x) := \sum_{\psi=1}^{d}\delta_{x,X(\psi)}\hat{U}_{0,t}\hat{U}_x|\psi\rangle\langle\psi|\hat{U}_x^\dagger\hat{U}_{t,0}$$

← B NInv Bi

Appendix B: Initialization

Initialization: the required control over initial conditions

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) \to P^{\underline{X}_n}_{\underline{t}_n|t_0}(\underline{x}_n \mid \text{initialization})$$

Appendix B: Initialization

Initialization: the required control over initial conditions

$$P_{\underline{t}_n}^{\underline{X}_n}(\underline{x}_n) \rightarrow P_{\underline{t}_n|t_0}^{\underline{X}_n}(\underline{x}_n \mid \text{initialization})$$

Markovianity: it seems that without it physics is impossible

$$P_{\underline{t}_n}^{X_0}(\underline{x}_n) = P_{t_1}^{X_0}(x_1) \prod_{k=1}^{n-1} P_{t_{k+1}|t_k}^{X_0|X_0}(x_{k+1} \mid x_k)$$

Appendix B: Initialization

Initialization: the required control over initial conditions

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) \to P^{\underline{X}_n}_{\underline{t}_n|t_0}(\underline{x}_n \mid \text{initialization})$$

Markovianity: it seems that without it physics is impossible

$$P^{X_0}_{\underline{t}_n}(\underline{x}_n) = P^{X_0}_{t_1}(x_1)\prod_{k=1}^{n-1} P^{X_0|X_0}_{t_{k+1}|t_k}(x_{k+1} \mid x_k) \Rightarrow P^{\underline{X}_n,X_0,X_{-1},X_{-2}\ldots}_{\underline{t}_n,t_0,t_{-1},t_{-2}\ldots}(\underline{x}_n, x_0, x_{-1}, x_{-2}\ldots) = P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0)P^{X_0,X_{-1}\ldots}_{t_0,t_{-1}\ldots}(x_0\ldots)$$

Appendix B: Initialization

Initialization: the required control over initial conditions

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) \to P^{\underline{X}_n}_{\underline{t}_n|t_0}(\underline{x}_n \mid \text{initialization})$$

Markovianity: it seems that without it physics is impossible

$$P^{X_0}_{\underline{t}_n}(\underline{x}_n) = P^{X_0}_{t_1}(x_1)\prod_{k=1}^{n-1} P^{X_0|X_0}_{t_{k+1}|t_k}(x_{k+1} \mid x_k) \Rightarrow P^{\underline{X}_n,X_0,X_{-1},X_{-2}\ldots}_{\underline{t}_n,t_0,t_{-1},t_{-2}\ldots}(\underline{x}_n, x_0, x_{-1}, x_{-2}\ldots) = P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0)P^{X_0,X_{-1}\ldots}_{t_0,t_{-1}\ldots}(x_0\ldots)$$

Measurement as initialization: “we observe correlations”, or “there is no absolute time”

$$P^{\underline{X}_n}_{\underline{t}_n|t_0}(\underline{x}_n \mid \text{initialization}) = P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0)$$

Appendix B: Initialization

Initialization: the required control over initial conditions

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) \to P^{\underline{X}_n}_{\underline{t}_n|t_0}(\underline{x}_n \mid \text{initialization})$$

Markovianity: it seems that without it physics is impossible

$$P^{X_0}_{\underline{t}_n}(\underline{x}_n) = P^{X_0}_{t_1}(x_1)\prod_{k=1}^{n-1} P^{X_0|X_0}_{t_{k+1}|t_k}(x_{k+1} \mid x_k) \quad \Rightarrow \quad P^{\underline{X}_n,X_0,X_{-1},X_{-2}\dots}_{\underline{t}_n,t_0,t_{-1},t_{-2}\dots}(\underline{x}_n, x_0, x_{-1}, x_{-2}\dots) = P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0) P^{X_0,X_{-1}\dots}_{t_0,t_{-1}\dots}(x_0\dots)$$

Measurement as initialization: "we observe correlations", or "there is no absolute time"

$$P^{\underline{X}_n}_{\underline{t}_n|t_0}(\underline{x}_n \mid \text{initialization}) = P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0) \quad \Rightarrow \quad P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \sum_{X_0,x_0} P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0) p^{X_0}(x_0)$$

$$\left(\sum_{X_0,x_0} p^{X_0}(x_0) = 1 \quad p^{X_0}(x_0) \geq 0\right)$$

← A Std NInv Bi

Appendix B: Initialization

Initialization: the required control over initial conditions

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) \to P^{\underline{X}_n}_{\underline{t}_n|t_0}(\underline{x}_n \mid \text{initialization})$$

Markovianity: it seems that without it physics is impossible

$$P^{X_0}_{\underline{t}_n}(\underline{x}_n) = P^{X_0}_{t_1}(x_1)\prod_{k=1}^{n-1} P^{X_0|X_0}_{t_{k+1}|t_k}(x_{k+1} \mid x_k) \quad \Rightarrow \quad P^{\underline{X}_n,X_0,X_{-1},X_{-2}\dots}_{\underline{t}_n,t_0,t_{-1},t_{-2}\dots}(\underline{x}_n, x_0, x_{-1}, x_{-2}\dots) = P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0) P^{X_0,X_{-1}\dots}_{t_0,t_{-1}\dots}(x_0\dots)$$

Measurement as initialization: “we observe correlations”, or “there is no absolute time”

$$P^{\underline{X}_n}_{\underline{t}_n|t_0}(\underline{x}_n \mid \text{initialization}) = P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0) \quad \Rightarrow \quad P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \sum_{X_0,x_0} P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0) p^{X_0}(x_0)$$

$$\left(\sum_{X_0,x_0} p^{X_0}(x_0) = 1 \quad p^{X_0}(x_0) \geq 0\right)$$

Density matrix: a convenient way for encoding initialization in bi-trajectory formalism

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \iint \left\{ \prod_{k=1}^{n} \delta_{x_k, X_k(\psi^+(t_k,\hat{U}_k))} \delta_{x_k, X_k(\psi^-(t_k,\hat{U}_k))} \right\} Q[\psi^+, \psi^-][\mathrm{D}\psi^+][\mathrm{D}\psi^-]$$

Appendix B: Initialization

Initialization: the required control over initial conditions

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) \to P^{\underline{X}_n}_{\underline{t}_n|t_0}(\underline{x}_n \mid \text{initialization})$$

Markovianity: it seems that without it physics is impossible

$$P^{X_0}_{\underline{t}_n}(\underline{x}_n) = P^{X_0}_{t_1}(x_1)\prod_{k=1}^{n-1} P^{X_0|X_0}_{t_{k+1}|t_k}(x_{k+1} \mid x_k) \quad \Rightarrow \quad P^{\underline{X}_n,X_0,X_{-1},X_{-2}\dots}_{\underline{t}_n,t_0,t_{-1},t_{-2}\dots}(\underline{x}_n, x_0, x_{-1}, x_{-2}\dots) = P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0)P^{X_0,X_{-1}\dots}_{t_0,t_{-1}\dots}(x_0\dots)$$

Measurement as initialization: "we observe correlations", or "there is no absolute time"

$$P^{\underline{X}_n}_{\underline{t}_n|t_0}(\underline{x}_n \mid \text{initialization}) = P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0) \quad \Rightarrow \quad P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \sum_{X_0,x_0} P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0)p^{X_0}(x_0) \qquad \left(\sum_{X_0,x_0} p^{X_0}(x_0) = 1 \quad p^{X_0}(x_0) \geq 0\right)$$

Density matrix: a convenient way for encoding initialization in bi-trajectory formalism

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \iint \left\{\prod_{k=1}^{n} \delta_{x_k, X_k(\psi^+(t_k,\hat{U}_k))}\delta_{x_k,X_k(\psi^-(t_k,\hat{U}_k))}\right\} Q[\psi^+,\psi^-][D\psi^+][D\psi^-]$$

$$\to \iint \left\{\prod_{k=1}^{n} \delta_{x_k, X_k(\psi^+(t_k,\hat{U}_k))}\delta_{x_k,X_k(\psi^-(t_k,\hat{U}_k))}\right\}\left\{\sum_{x_0} p^{X_0}(x_0)\delta_{x_0,X_0(\psi^+(t_0,\hat{U}_{X_0}))}\delta_{x_0,X_0(\psi^-(t_0,\hat{U}_{X_0}))}\right\} Q[\psi^+,\psi^-][D\psi^+][D\psi^-]$$

Appendix B: Initialization

Initialization: the required control over initial conditions

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) \to P^{\underline{X}_n}_{\underline{t}_n|t_0}(\underline{x}_n \mid \text{initialization})$$

Markovianity: it seems that without it physics is impossible

$$P^{X_0}_{\underline{t}_n}(\underline{x}_n) = P^{X_0}_{t_1}(x_1)\prod_{k=1}^{n-1} P^{X_0|X_0}_{t_{k+1}|t_k}(x_{k+1} \mid x_k) \quad \Rightarrow \quad P^{\underline{X}_n,X_0,X_{-1},X_{-2}\ldots}_{\underline{t}_n,t_0,t_{-1},t_{-2}\ldots}(\underline{x}_n, x_0, x_{-1}, x_{-2}\ldots) = P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0)P^{X_0,X_{-1}\ldots}_{t_0,t_{-1}\ldots}(x_0\ldots)$$

Measurement as initialization: "we observe correlations", or "there is no absolute time"

$$P^{\underline{X}_n}_{\underline{t}_n|t_0}(\underline{x}_n \mid \text{initialization}) = P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0) \quad \Rightarrow \quad P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \sum_{X_0,x_0} P^{\underline{X}_n|X_0}_{\underline{t}_n|t_0}(\underline{x}_n \mid x_0)p^{X_0}(x_0)$$

$$\left(\sum_{X_0,x_0} p^{X_0}(x_0) = 1 \quad p^{X_0}(x_0) \geq 0\right)$$

Density matrix: a convenient way for encoding initialization in bi-trajectory formalism

$$P^{\underline{X}_n}_{\underline{t}_n}(\underline{x}_n) = \iint \left\{\prod_{k=1}^{n} \delta_{x_k,X_k(\psi^+(t_k,\hat{U}_k))}\delta_{x_k,X_k(\psi^-(t_k,\hat{U}_k))}\right\}Q[\psi^+,\psi^-][D\psi^+][D\psi^-]$$

$$\to \iint \left\{\prod_{k=1}^{n} \delta_{x_k,X_k(\psi^+(t_k,\hat{U}_k))}\delta_{x_k,X_k(\psi^-(t_k,\hat{U}_k))}\right\}\left\{\sum_{x_0} p^{X_0}(x_0)\delta_{x_0,X_0(\psi^+(t_0,\hat{U}_{X_0}))}\delta_{x_0,X_0(\psi^-(t_0,\hat{U}_{X_0}))}\right\}Q[\psi^+,\psi^-][D\psi^+][D\psi^-]$$

$$\hat{X}_0 = \sum_{\psi=1}^{d} X_0(\psi)\hat{U}_{X_0}|\psi\rangle\langle\psi|\hat{U}^\dagger_{X_0}$$

← A Std NInv Bi

Appendix B: Initialization

Initialization: the required control over initial conditions

$$P_{\underline{t}_n}^{\underline{X}_n}(\underline{x}_n) \to P_{\underline{t}_n|t_0}^{\underline{X}_n}(\underline{x}_n \mid \text{initialization})$$

Markovianity: it seems that without it physics is impossible

$$P_{\underline{t}_n}^{X_0}(\underline{x}_n) = P_{t_1}^{X_0}(x_1)\prod_{k=1}^{n-1} P_{t_{k+1}|t_k}^{X_0|X_0}(x_{k+1} \mid x_k) \quad \Rightarrow \quad P_{\underline{t}_n,t_0,t_{-1},t_{-2}\dots}^{\underline{X}_n,X_0,X_{-1},X_{-2}\dots}(\underline{x}_n, x_0, x_{-1}, x_{-2}\dots) = P_{\underline{t}_n|t_0}^{\underline{X}_n|X_0}(\underline{x}_n \mid x_0)P_{t_0,t_{-1}\dots}^{X_0,X_{-1}\dots}(x_0\dots)$$

Measurement as initialization: "we observe correlations", or "there is no absolute time"

$$P_{\underline{t}_n|t_0}^{\underline{X}_n}(\underline{x}_n \mid \text{initialization}) = P_{\underline{t}_n|t_0}^{\underline{X}_n|X_0}(\underline{x}_n \mid x_0) \quad \Rightarrow \quad P_{\underline{t}_n}^{\underline{X}_n}(\underline{x}_n) = \sum_{X_0,x_0} P_{\underline{t}_n|t_0}^{\underline{X}_n|X_0}(\underline{x}_n \mid x_0)p^{X_0}(x_0) \qquad \left(\sum_{X_0,x_0} p^{X_0}(x_0) = 1 \quad p^{X_0}(x_0) \geq 0\right)$$

Density matrix: a convenient way for encoding initialization in bi-trajectory formalism

$$P_{\underline{t}_n}^{\underline{X}_n}(\underline{x}_n) = \iint \Big\{\prod_{k=1}^{n} \delta_{x_k,X_k(\psi^+(t_k,\hat{U}_k))}\delta_{x_k,X_k(\psi^-(t_k,\hat{U}_k))}\Big\}Q[\psi^+,\psi^-][D\psi^+][D\psi^-]$$

$$\to \iint \Big\{\prod_{k=1}^{n} \delta_{x_k,X_k(\psi^+(t_k,\hat{U}_k))}\delta_{x_k,X_k(\psi^-(t_k,\hat{U}_k))}\Big\}\Big\{\sum_{x_0} p^{X_0}(x_0)\delta_{x_0,X_0(\psi^+(t_0,\hat{U}_{X_0}))}\delta_{x_0,X_0(\psi^-(t_0,\hat{U}_{X_0}))}\Big\}Q[\psi^+,\psi^-][D\psi^+][D\psi^-]$$

$$\hat{X}_0 = \sum_{\psi=1}^{d} X_0(\psi)\hat{U}_{X_0}|\psi\rangle\langle\psi|\hat{U}_{X_0}^\dagger$$

$$\sum_{\psi=1}^{d}\sum_{X_0,x_0} p^{X_0}(x_0)\delta_{x_0,X_0(\psi)}\hat{U}_{X_0}|\psi\rangle\langle\psi|\hat{U}_{X_0}^\dagger = \sum_{n} r_n|\Psi_n\rangle\langle\Psi_n| = \hat{\rho}_{t_0}$$

← A Std NInv Bi

Abstract

Even among specialists, quantum mechanics is notorious for being difficult—or even impossible—to understand. The standard approach to this problem is to explain the numerous idiosyncrasies of quantum theory by comparing and contrasting it with classical theories. This tried-and-true strategy relies, of course, on our solid understanding of classical theories—contrasting the classical with the quantum can shed light on the latter only if there are no doubts about the former. However, is our confidence in classical physics truly justified?

In this talk, I will present a critical examination of our current understanding of classical theories in the context of their comparison with quantum mechanics. I will argue that this understanding is far from complete, and that much remains to be learned. Finally, I will outline a systematic strategy for addressing this issue and show that many classical features can, in fact, be best explained through comparison with quantum mechanics—an ironic turn of events, given the context.